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DETERMINATION OF THE NORMAL FORMS OF THE HAMILTONIAN MATRICES*

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A method of finding the generating function of a canonical transformation reducing a quadratic Hamltonian and the corresponding Hamiltonian matrix to some normal form, is obtained. The problem of reducing a fourth order Hamiltonian matrix to its normal form is solved as an example.

Let us consider a canonical system of differential equations with quadratic Hamiltonian

$$dx/dt = \partial H/\partial y,\ dy/dt = -\partial H/\partial x,\ \ H(x, y) = {}^1/_2 y'Cy + x'By + {}^1/_2 x'Ax \tag{1}$$

Here x and y are n-dimensional column vectors of the canonical conjugate variables, A, B and C are real, n-th order square matrices, A and C are symmetric matrices and a prime denotes transposition. The system (1) can also be written in the form

$$\frac{d}{dt}\left\|\begin{matrix} x \\ y \end{matrix}\right\| = V\left\|\begin{matrix} x \\ y \end{matrix}\right\|,\quad V = \left\|\begin{matrix} B' & C \\ -A & -B \end{matrix}\right\| \tag{2}$$

where V is a Hamiltonian matrix.

The method of normalizing an arbitrary Hamiltonian matrix /1/ in order to find a normalizing canonical transformation is not very practical. Other methods were therefore developed in /2−5/ where the authors imposed various constraints on the Hamiltonian matrix (in particular, the nondegeneracy of the Hamiltonian matrix was assumed in all cases). Below we give a method of obtaining a generating function of a canonical variable transformation which transforms the Hamiltonian matrix to some standard form. The matrix in this case may have multiple and zero eigenvalues.

Let q and p be n-dimensional column vectors of the new canonical variables. We carry out the canonical transformation with help of the generating function

$$S(x, p) = {}^1/_2 p'Kp + p'Lx + {}^1/_2 x'Mx \tag{3}$$

Here K, L and M are n-th order square matrices, K and M are symmetric matrices and L is a nondegenerate matrix. In addition, the equations

$$\partial S/\partial p = q,\ \partial S/\partial x = y$$

yield the following formula expressing the old variables in terms of the new variables:

$$\left\|\begin{matrix} x \\ y \end{matrix}\right\| = \left\|\begin{matrix} L^{-1} & -L^{-1}K \\ ML^{-1} & L' - ML^{-1}K \end{matrix}\right\| \left\|\begin{matrix} q \\ p \end{matrix}\right\| \tag{4}$$

Carrying out the necessary calculations, we obtain the new Hamiltonian and use the obvious identity

$$2q'BMq = q'(BM + MB')q$$

to reduce it to the following form:

$$\begin{aligned} &H(q, p) = {}^1/_2 p'C_0 p + q'B_0 p + {}^1/_2 q'A_0 q \\ &A_0 = A_0' = (L')^{-1}(MCM + MB' + BM + A)L^{-1} \\ &B_0 = (L')^{-1}(MC + B)L' - A_0 K \\ &C_0 = C_0' = -KA_0K - KB_0 - B_0'K + LCL' \end{aligned}$$

Thus, if we find it necessary to reduce the canonical system with help of a canonical transformation of the form (4) to a system with the Hamiltonian matrix

$$V_0 = \left\|\begin{matrix} B_0' & C_0 \\ -A_0 & -B_0 \end{matrix}\right\| \tag{5}$$

*Prikl.Matem.Mekhan.,45,No.6,1026-1031,1981 then the unknown matrices K, L and M of the generating function must be sought from the following system of matrix equations:

$$\begin{aligned} &MCM + MB' + BM + A = L'A_0L \\ &L(CM + B')L^{-1} = KA_0 + B_0' \\ &KA_0K + KB_0 + B_0'K + C_0 = LCL' \end{aligned} \tag{6}$$

The matrices A, A_0, C, C_0 in this system are symmetric, the unknown matrices M and K are sought in the symmetric form and the matrix L in the nondegenerate form. Using the system (6) we can reduce a Hamiltonian matrix to its normal form, pass from one normal form to another, and simplify the initial Hamiltonian matrix.

Let us assume it necessary to reduce a Hamiltonian matrix to the following normal form:

$$\Phi = \begin{Vmatrix} U & I \\ O & -U' \end{Vmatrix}, \quad U = \begin{Vmatrix} \lambda_1 & \varepsilon_1 & 0 & \ldots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \varepsilon_{n-1} \\ 0 & \cdot & \cdot & \cdot & \lambda_n \end{Vmatrix}, \quad I = \operatorname{diag}\{\varepsilon_n, \ldots, \varepsilon_{2n-1}\}, \quad \varepsilon_i = \begin{cases} 0, 1; & i = 1, \ldots, n-1 \\ 0, \pm 1; & i = n, \ldots, 2n-1 \end{cases} \tag{7}$$

where λ_i are the eigenvalues of the Hamiltonian matrix V. The corresponding Hamiltonian has the form

$$H(q, p) = {}^1/_2 p'Ip + p'Uq = \sum_{i=1}^{n} ({}^1/_2\varepsilon_{n+i-1}p_i^2 + \lambda_i q_i p_i) + \sum_{i=1}^{n-1} \varepsilon_i p_i q_{i+1}$$

Then the sysyem (6) assumes the form

$$\begin{aligned} &MCM + MB' + BM + A = 0 \\ &L(CM + B')L^{-1} = U, \; KU' + UK = LCL' - I \end{aligned} \tag{8}$$

and we find the unknown matrices for this system one after the other. The first equation, which is a matrix Riccati equation /6,7/, yields the symmetric matrix M, the second equation yields L which transforms the matrix $CM + B'$ to the normal Jordan form, and the third equation yields the symmetric matrix K.

Assertion 1. The canonical transformation defined by a generating function of the form (3) where the matrix L is nondegenerate, reduces the Hamiltonian matrix V of the system (2) to the normal form (7) if and only if the matrices K, L and M of the generating function are solutions of the system of matric equations (8).

Assertion 2. The system (8) of matrix equations has a solution if and only if a symplectic matrix T of order $2n$ exists such, that $T^{-1}VT = \Phi$, and the elements of the matrix T situated at the intersections of the first n columns and n rows form a nondegenerate matrix.

Proof. The necessity follows from the fact that the transformation matrix (4) satisfies the conditions of the assertion, and we shall prove the sufficiency. Let the matrix T have the form

$$T = \begin{Vmatrix} F & H \\ G & M \end{Vmatrix} \tag{9}$$

Here F, G, H and W are n-th order square matrices, and $\det F \neq 0$. Then it can be shown that the following matrices are solutions of the system (8):

$$M = GF^{-1}, \; L = F^{-1}, \; K = -F^{-1}H$$

Assertion 3. Let a solution of the first equation of (8), i.e. of the matrix Riccati equation, exist. Then a matrix U of normal Jordan form and a some symmetric matrix I (not necessarily diagonal) for which the system (8) has a solution, both exist.

Proof. Knowing the solution of the first equation of (8), we can indeed find the matrices U and I in the course of solving the remaining equations. The matrix U represents the normal Jordan form of the matrix $CM + B'$, and the matrix I ensures the solvability of the third equation of the system. For example, when $I = LCL'$, then the third equation has the solution $K = 0$.

Thus we see that the solution of (8) depends mainly on the solution of the matrix Riccati equation. Various methods of finding a symmetric solution of this nonlinear equation exist /6 — 8/, including the numerical. We shall consider one of these methods. Let T be an arbitrary matrix of the form (9), reducing the Hamiltonian matrix to the normal form (7) in such a manner that $T^{-1}VT = \Phi$, and $\det F \neq 0$. Then the first two equations of (8) will have the following solutions /6,7/:

$$M = GF^{-1},\ L = F^{-1}$$

The symmetric character of the matrix GF^{-1} is shown in /6,7/ under the condition that the eigenvalues of the matrix U, which appears in the normal type matrix Φ, satisfy the condition $\lambda_i + \lambda_j \neq 0,\ i, j = 1, \ldots, n$. The condition will always hold, provided that the Hamiltonian matrix V is nondegenerate. Let $\det V \neq 0$, k_{ij} be the elements of the matrix K, and c_{ij} the elements of the matrix LCL' (we note that $I = 0$ since $\det V \neq 0$). Then the elements of the symmetric matrix K are given by the triangular system of $\frac{1}{2}n\cdot(n+1)$ linear equations

$$(\lambda_i + \lambda_j)\, k_{ij} + \varepsilon_i k_{i+1,j} + \varepsilon_j k_{i,j+1} = c_{ij} \qquad (10)$$
$$1 \leqslant i \leqslant j \leqslant n,\ k_{ij} = k_{ji},\ \varepsilon_n = k_{n+1,j} = 0$$

The system is compatible, since the principal determinant of the system

$$\prod_{1\leqslant i\leqslant j\leqslant n} (\lambda_i + \lambda_j) \neq 0$$

Assertion 4. Let the Hamiltonian matrix V have at most a single pair of zero eigenvalues, T be an arbitrary matrix of the form (9) reducing the matrix V to the normal form (7), and let the following conditions hold:

1) the matrices U and I appearing in the normal type matrix Φ have the form

$$U = \begin{Vmatrix} \lambda_1 & \varepsilon_1 & 0 \ldots 0 \\ 0 & \lambda_2 & \varepsilon_2 \ldots 0 \\ . & . & . \quad . \\ 0 & . & . \quad \lambda_n \end{Vmatrix}, \quad I = \operatorname{diag}\{0, \ldots, 0, \varepsilon\}$$
$$\lambda_i + \lambda_j \neq 0;\ 2 \leqslant i + j \leqslant 2n - 1;\ \varepsilon_i = 0,1;\ i = 1, \ldots, n-1;\ \varepsilon = 0,1$$

(the above condition can always be realized under the conditions imposed on the eigenvalues of V /9/);

2) the transformation matrix T has a nondegenerate submatrix F. Then the matrices of the generating function of some normalizing canonical transformation have the form

$$M = GF^{-1}, \quad L = lF^{-1}, \quad l = \begin{cases} 1/\sqrt{f}, & f \neq 0 \\ 1, & f = 0 \end{cases}$$

where f is the element of the matrix $F^{-1}C(F^{-1})'$ appearing in the lower right-hand side corner. The elements of the symmetric matrix K are found from the compatible triangular system of linear equations (10) where c_{ij} are the elements of the matrix $LCL' - I$.

Proof. It whas shown (*) that the matrix $M = GF^{-1}$ is, under the conditions of the assertion, a symmetric solution of the matrix Riccati equation. It is clear that the matrix $L = lF^{-1}$ satisfies the second equation of (8). We shall show that the equations (10) are compatible. Indeed, the last equation of this system has the form

$$2\lambda_n k_{nn} = c_{nn},\ c_{nn} = l^2 f - \varepsilon$$

($\varepsilon = 0$ if the Hamiltonian matrix is nondegenerate). Out of the two values which ε can assume, we can always choose the value for which the equation has a solution (the matrices F and G are independent of ε). The remaining equations of the system (10) form a system of linear equations the principal determinant of which is not zero. Having found the generating function of the canonical transformation, we obtain the transformation itself in the form

$$\begin{Vmatrix} x \\ y \end{Vmatrix} = \frac{1}{l} \begin{Vmatrix} F & -FK \\ G & l^2 (F^{-1})' - GK \end{Vmatrix} \begin{Vmatrix} q \\ p \end{Vmatrix}$$

*) Titova T.N. On the normalizing of a degenerate Hamiltonian matrix. Moscow, Dep. v VINITI, No.434, 1978.

Example 1. We illustrate the proposed method by obtaining the expressions for the generating function of canonical transformation normalizing the nondegenerate fourth order Hamiltonian matrix V (*) under the condition that the submatrix C is positive definite (i.e. the corresponding Hamiltonian is a positive definite quadratic form relative to the generalized impulses). Then a nondegenerate matrix R exists such, that $R'CR=E$ /10/. To simplify the formulas, we shall first perform the following canonical transformation:

$$\begin{Vmatrix} x \\ y \end{Vmatrix} = \begin{Vmatrix} (R')^{-1} & O \\ O & R \end{Vmatrix} \begin{Vmatrix} \xi \\ \eta \end{Vmatrix}$$

Then the new Hamiltonian will become

$$H(\xi,\eta) = {}^1\!/_2\, \eta'\eta + \xi' R^{-1} B R \eta + {}^1\!/_2\, \xi' R^{-1} A (R^{-1})' \xi = {}^1\!/_2\, \eta'\eta + \xi' B_0 \eta + {}^1\!/_2\, \xi' A_0 \xi$$

In what follows, we shall omit the indices and consider the following Hamiltonian matrix:

$$V = \begin{Vmatrix} B' & E \\ -A & -B \end{Vmatrix} = \begin{Vmatrix} b_{11} & b_{21} & 1 & 0 \\ b_{12} & b_{22} & 0 & 1 \\ -a_{11} & -a_{12} & -b_{11} & -b_{12} \\ -a_{12} & -a_{22} & -b_{21} & -b_{22} \end{Vmatrix}$$

Let $\mu = b_{12} - b_{21}$, $D = BB' - A$, ν_1 and ν_2 be the eigenvalues of the matrix D, $\pm\lambda_1$, $\pm\lambda_2$ the eigenvalues of V, $\sqrt{\nu_i}$ a positive or purely imaginary number with a positive coefficient, and F an orthogonal matrix reducing the symmetric matrix D to the diagonal form, i.e. $FDF' = \operatorname{diag}\{\nu_1, \nu_2\}$ (the rows of the matrix F are orthogonal eigenvectors of the matrix D).

We shall reduce the Hamiltonian matrix V to the normal form

$$\Phi = \begin{Vmatrix} U & O \\ O & -U' \end{Vmatrix}$$

Case 1°. Let $\mu = \pm(\sqrt{\nu_1} + \sqrt{\nu_2})$; $\nu_1 \neq \nu_2$. Then

$$\lambda_1 = \lambda_2 = \lambda; \quad \lambda^2 = -\sqrt{\nu_1 \nu_2},$$

$$U = \begin{Vmatrix} \lambda & 1 \\ 0 & \lambda \end{Vmatrix}, \quad p = 1/(\sqrt{\nu_1} - \sqrt{\nu_2})$$

$$M = pF' \begin{Vmatrix} \pm 2\lambda\sqrt{\nu_1} & \sqrt{\nu_1}\mu \\ \sqrt{\nu_2}\mu & \mp 2\lambda\sqrt{\nu_2} \end{Vmatrix} F - B$$

$$L = \begin{Vmatrix} 0 & 1 \\ \sqrt{\nu_1}\mu p & \lambda \mp 2\lambda\sqrt{\nu_1}p \end{Vmatrix} F$$

$$k_{22} = (\lambda^2 \mp 4\lambda^2\sqrt{\nu_1}p + \nu_1)/(2\lambda), \quad k_{12} = k_{21} = (\lambda \mp 2\lambda\sqrt{\nu_1}p - k_{22})/(2\lambda),$$

$$k_{11} = (1 - 2k_{12})/(2\lambda)$$

Case 2°. Let $\mu = \pm(\sqrt{\nu_1} - \sqrt{\nu_2})$; $\nu_1 \neq \nu_2$. All formulas given in 1° hold after replacing $\sqrt{\nu_2}$ by $-\sqrt{\nu_2}$.

Case 3°. Let $\mu^2 \neq (\sqrt{\nu_1} \pm \sqrt{\nu_2})^2$; $\mu \neq 0$. Then the eigenvalues of the matrix V will be different: $\pm\lambda_1$, $\pm\lambda_2$; $U = \operatorname{diag}\{\lambda_1, \lambda_2\}$.

$$q = \mu/(\sqrt{\nu_1} + \sqrt{\nu_2}), \quad r = \pm\sqrt{1 - q^2}$$

$$M = F' \begin{Vmatrix} \sqrt{\nu_1}r & \sqrt{\nu_1}q \\ -\sqrt{\nu_2}q & \sqrt{\nu_2}r \end{Vmatrix} F - B$$

$$L = \begin{Vmatrix} \sqrt{\nu_1}q & \lambda_1 - \sqrt{\nu_1}r \\ \sqrt{\nu_1}q & \lambda_2 - \sqrt{\nu_1}r \end{Vmatrix} F$$

$$k_{11} = (\nu_1 + \lambda_1^2 - 2\lambda_1\sqrt{\nu_1}r)/(2\lambda_1)$$

$$k_{12} = k_{21} = (\nu_1 + \lambda_1\lambda_2 - (\lambda_1 + \lambda_2)\sqrt{\nu_1}r)/(\lambda_1 + \lambda_2)$$

$$k_{22} = (\nu_1 + \lambda_2^2 - 2\lambda_2\sqrt{\nu_1}r)/(2\lambda_2)$$

Case 4°. Let $\mu = 0$, $\nu_1 \neq \nu_2$. Then $U = \operatorname{diag}\{\sqrt{\nu_1}, \sqrt{\nu_2}\}$, $M = F'UF - B$, $L = F$, $K = {}^1\!/_2\, U^{-1}$.

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*) Titova T.N. On the normalization of a linear Hamiltonian system with help of canonical transformations. Moscow, Dep. v. VINITI, No. 1049, 1976.

Case 5°. Let $\mu = 0, \nu_1 = \nu_2 = \nu$. Then $U = \sqrt{\nu}E, M = \sqrt{\nu}E - B, L = E, \quad K = (1/(2\sqrt{\nu}))E$.

Thus the method fails when $\nu_1 = \nu_2 = \mu^2/4$.

Example 2. Consider a Hamiltonian matrix of the type (7) where

$$U = U_k \dotplus O_{n-k}, \quad I = O_k \dotplus I_{n-k}, \quad U_k = \operatorname{diag}\{\lambda_1, \ldots, \lambda_k\}, \quad I_{n-k} = \operatorname{diag}\{\varepsilon_1, \ldots, \varepsilon_{n-k}\}, \quad \varepsilon_i = 0, \ \pm 1$$

and O_k (is a zero square matrix of order k ($\dotplus$ denotes a straight sum of the matrices). Any Hamiltonian matrix the nonzero eigenvalues of which do not form Jordanian cells of order higher than the first, can be reduced to such normal form. Let all λ_i $(i = 1, \ldots, k)$ be purely imaginary. Then the Hamiltonian matrix is complex and we have the problem of passing to the following real form:

$$\begin{Vmatrix} O & C \\ -A & O \end{Vmatrix}, \quad C = -iU_k \dotplus I_{n-k}, \quad A = -iU_k \dotplus O_{n-k}$$

The system of matrix equations (6) assumes in this case the following form:

$$MIM + MU + UM = L'AL$$
$$L(IM + U)L^{-1} = KA, \quad KAK + C = LIL'$$

and this yields the matrices of the generating function

$$M = -{}^1/_2 iE_k \dotplus O_{n-k}, L = E, K = iE_k \dotplus O_{n-k}$$

The canonical transformation is obtained in the form

$$x_1 = q_1 - ip_1, \quad x_2 = q_2 - ip_2, \ldots, \quad x_k = q_k - ip_k, \quad x_{k+1} = q_{k+1}, \ldots,$$
$$x_n = q_n, \quad y_1 = -{}^1/_2 iq_1 + {}^1/_2 p_1, \ldots, y_k = -{}^1/_2 iq_k + {}^1/_2 p_k, y_{k+1} = p_{k+1}, \ldots, y_n = p_n$$

and putting $\lambda_j = i\omega_j$ we obtain, as the result of the transformation, the new Hamiltonian in the form

$$H(q, p) = \frac{1}{2} \sum_{j=1}^{k} \omega_j (p_j^2 + q_j^2) + \frac{1}{2} \sum_{j=k+1}^{n} p_j^2$$

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